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A canonical transformation of the Hamiltonians quadratic in coordinate and momentum operators

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Abstract. In this paper an algebraic (matrix) method is presented for canonically transforming the Hamiltonians quadratic in coordinate and momentum operators into the Hamiltonian of non-interacting harmonic oscillators. The method is illustrated by transforming the Hamiltonian of a harmonic oscillator in a constant magnetic field into the Hamiltonian of the two-dimensional, in general anisotropic, harmonic oscillator. The results are found to be in agreement with those obtained previously for the same Hamiltonian using a different technique.

1. Introduction

After long neglect, interest in canonical transformations has recently been revived, following the fundamental work of Moshinsky and Quesne (1971).

In a number of applications, canonical transformations have been used to relate certain Hamiltonians to the Hamiltonian of a harmonic oscillator; for example, the two-dimensional Coulomb Hamiltonian (Moshinsky *et al* 1972), the Hamiltonian of an electron in a uniform magnetic field (Boon and Seligman 1973) and the time-dependent Hamiltonian quadratic in coordinate and momentum operators (Leach 1977, 1978).

In this paper, we consider a general time-independent Hamiltonian quadratic in coordinate and momentum operators. In § 2, a method is given for canonically transforming this quadratic Hamiltonian into the Hamiltonian of the independent harmonic oscillators. It is found that these harmonic oscillator Hamiltonians belong to three distinct types: (i) the attractive harmonic oscillator, (ii) the repulsive harmonic oscillator and (iii) the two-dimensional isotropic harmonic oscillator with a term proportional to the angular momentum of a particle moving in the plane. These three types of transformed Hamiltonian can possibly be related to the bound and unbound (scattered) states of the original Hamiltonian. The results of this section are equally applicable to classical and quantum systems.

In § 3, the method is illustrated by canonically transforming the Hamiltonian of a harmonic oscillator in a constant magnetic field into the Hamiltonian of the two-dimensional (in general, anisotropic) harmonic oscillator. The same Hamiltonian has been studied by Dulock and McIntosh (1966) and their results are in agreement with those obtained in this paper by applying the general method.

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The limits of the method's applicability, the possible meaning of the results and the outline for future developments are briefly mentioned in § 4.

2. Method

2.1. Notation and statement of the problem

The $2n \times 2n$ symplectic matrix is introduced as

$$\mathbf{K} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} \tag{2.1}$$

where $\mathbf{I}$ is the $n \times n$ unit matrix and n is a positive integer. In addition to being symplectic the matrix $\mathbf{K}$ (2.1) is orthogonal:

$$\mathbf{K}\tilde{\mathbf{K}} = \mathbf{I} \tag{2.2}$$

where the matrix $\tilde{\mathbf{K}}$ is the transpose of the matrix $\mathbf{K}$ and it is skew-symmetric

$$\mathbf{K} = -\tilde{\mathbf{K}}. \tag{2.3}$$

The system of n degrees of freedom is described by the set of n coordinates $x_1 \ldots x_n$, which are understood to be components of a vector $\boldsymbol{x}$, and the set of n momenta $p_1 \ldots p_n$, the components of the vector $\boldsymbol{p}$. These two vectors define a state of the system as a point in $2n$-dimensional phase space. The linear canonical transformation changes the vectors $\boldsymbol{x}$ and $\boldsymbol{p}$ into the new ones:

$$\begin{pmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{p}} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix} \tag{2.4}$$

preserving the commutational relations between coordinate and momentum operators (or the Poisson brackets in the classical case)

$$[x_\alpha, x_\beta] = 0, \qquad [\chi_\alpha, p_\beta] = \mathrm{i}\delta_{\alpha\beta} \qquad \text{and} \qquad [p_\alpha, p_\beta] = 0. \tag{2.5}$$

It can be shown (Moshinsky and Quesne 1971) that the transformation effected by the matrix $\mathbf{M}$ is canonical if

$$\mathbf{M}\mathbf{K}\tilde{\mathbf{M}} = K. \tag{2.6}$$

Also, it follows from equation (2.6) that if the matrix $\mathbf{M}$ is canonical so is $\mathbf{M}^{-1}$, and if two matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are canonical, the product $\mathbf{M}_1\mathbf{M}_2$ is also a canonical matrix, i.e. the set of matrices satisfying equation (2.6) form a group—the symplectic group of order $2n$, $\mathrm{Sp}(2n)$.

The object of the study of this paper is the general Hamiltonian quadratic in coordinate and momentum operators

$$\mathscr{H} = \sum_{\alpha\beta} H^{(1)}_{\alpha\beta} x_\alpha x_\beta + \sum_{\gamma\delta} H^{(2)}_{\gamma\delta} x_\gamma p_\delta + \sum_{\zeta\eta} H^{(3)}_{\zeta\eta} p_\zeta x_\eta + \sum_{\mu\nu} H^{(4)}_{\mu\nu} p_\mu p_\nu \tag{2.7}$$

where the indices are summed from 1 to n. The coefficients $H^{(i)}_{\alpha\beta}$, $i = 1 \ldots 4$ are all real and can always be made symmetric by using the relation

$$x_\alpha p_\beta = \tfrac{1}{2}(x_\alpha p_\beta + p_\beta x_\alpha). \tag{2.8}$$

In the vector notation the Hamiltonian (2.7) is

$$\mathcal{H}=\begin{pmatrix}x\\ p\end{pmatrix}\mathbf{H}\begin{pmatrix}x\\ p\end{pmatrix} \tag{2.9}$$

where the matrix **H** has elements $H^{(i)}_{\alpha\beta}$, $i=1\ldots4$; $\alpha,\beta=1\ldots n$.

The problem is now formulated as follows. For a given Hamiltonian and the Hamiltonian matrix **H**, find a matrix **F** which diagonalises the Hamiltonian matrix **H**

$$\tilde{\mathbf{F}}\mathbf{H}\mathbf{F}=\mathbf{D}_{\mathrm{H}} \tag{2.10a}$$

(where $\mathbf{D}_{\mathrm{H}}$ is a diagonal matrix) subject to the condition of canonicity

$$\mathbf{F}\mathbf{K}\tilde{\mathbf{F}}=\mathbf{K}. \tag{2.10b}$$

The solution of the two matrix equations (2.10*a*) and (2.10*b*) for unknown matrices **F** and **D** is carried out in two steps as follows. First, in § 2.2, the canonical matrix **P** is found, which brings the Hamiltonian matrix **H** into pseudo-diagonal form

$$\tilde{\mathbf{P}}\mathbf{H}\mathbf{P}=\mathbf{H}^{1}=\tilde{\mathbf{K}}\mathbf{D} \tag{2.11}$$

where **D** is a diagonal matrix. Second, in § 2.3, a canonical matrix **T** is found which diagonalises the pseudo-diagonal matrix $\mathbf{H}^1$ and is independent of the elements of $\mathbf{H}^1$

$$\tilde{\mathbf{T}}\mathbf{H}^{1}\mathbf{T}=\mathbf{D}_{\mathrm{H}} \tag{2.12}$$

where $\mathbf{D}_{\mathrm{H}}$ is again a diagonal matrix. The solution to the problem is then the product of the matrices

$$\mathbf{F}=\mathbf{P}\mathbf{T}. \tag{2.13}$$

2.2. Pseudodiagonalisation of the Hamiltonian matrix

From the condition of canonicity for matrix **P**,

$$\mathbf{P}\mathbf{K}\tilde{\mathbf{P}}=\mathbf{K} \tag{2.14}$$

one can find the inverse of matrix $\tilde{\mathbf{P}}$;

$$\tilde{\mathbf{P}}^{-1}=\tilde{\mathbf{K}}\mathbf{P}\mathbf{K} \tag{2.15}$$

where the orthogonality of matrix **K** equation (2.2), has been used. Multiplying equation (2.11) by matrix $\tilde{\mathbf{P}}^{-1}$ from the left and using equation (2.15), one arrives at the equation

$$\mathbf{H}\mathbf{P}=\tilde{\mathbf{K}}\mathbf{P}\mathbf{D}. \tag{2.16}$$

The at present unspecified elements of the diagonal matrix **D** are denoted by $\lambda_1\ldots\lambda_{2n}$. The set of $2n\times2n$ equations contained in the matrix equation (2.16) for elements of matrix **P** separates into $2n$ equations for $2n$ columns of matrix **P**:

$$\sum_{\eta}H_{\mu\eta}P_{\eta\nu}=-\sum_{\tau\xi}K_{\mu\tau}P_{\tau\xi}\delta_{\xi\nu}$$
$$=\begin{cases}-\lambda_\nu P_{\mu+n\nu} & \text{for } \mu=1\ldots n\\ \lambda_\nu P_{\mu-n\nu} & \text{for } \mu=n+1\ldots2n\end{cases} \tag{2.17}$$

where $\nu = 1 \ldots 2n$. The elements of the matrix **K** (equation (2.1)) can be written as follows:

$$K_{\mu\nu} = \delta_{\mu\nu-n} \qquad \nu = n+1 \ldots 2n;$$

and

$$K_{\mu\nu} = -K_{\nu\mu} \qquad \nu, \mu = 1 \ldots 2n.$$

The homogeneous system of equations (2.17) has non-trivial solutions only if $\lambda_1 \ldots \lambda_{2n}$ are roots of the pseudosecular equation

$$\mathrm{d}(\mathbf{H} - \lambda \tilde{\mathbf{K}}) = 0 \tag{2.18}$$

where d(...) is a determinant of the enclosed matrix. The roots of equation (2.18) are symmetrically distributed around zero. Because a determinant does not change value upon transposition, it follows that

$$\tilde{\mathrm{d}}(\mathbf{H} - \lambda \tilde{\mathbf{K}}) = \mathrm{d}(\tilde{\mathbf{H}} - \lambda \mathbf{K}) = \mathrm{d}(\mathbf{H} + \lambda \tilde{\mathbf{K}}) = \mathrm{d}[\mathbf{H} - (-\lambda)\tilde{\mathbf{K}}] \tag{2.19}$$

where the symmetry of matrix **H** and the skew-symmetry of matrix **K** have been used. Therefore, if λ is a root of equation (2.18), $-\lambda$ is also a root.

The roots of equation (2.18) can be arranged in a sequence so that the pair of roots with opposite sign are n places apart:

$$\lambda_1, \lambda_2 \ldots \lambda_n, \; -\lambda_1, \; -\lambda_2 \ldots -\lambda_n. \tag{2.20}$$

In addition, all the roots are assumed to be different from zero and different from each other (no degeneracy).

The system of equations (2.17) can be written in a more compact form as a pseudo-eigenvalue problem for the columns of the matrix **P**:

$$\mathbf{H}(\boldsymbol{P}_\nu) = \lambda_\nu \tilde{\mathbf{K}}(\boldsymbol{P}_\nu) \qquad \nu = 1 \ldots 2n \tag{2.21}$$

where $(\boldsymbol{P}_\nu)$ is a column vector with elements $P_{1\nu} \ldots P_{2n\nu}$.

The system of equations (2.17) and (2.21) always has non-trivial solutions, provided $\lambda_1 \ldots \lambda_{2n}$ are the roots of equation (2.18) and the set of $2n$ vectors is linearly independent (Marcus and Ming 1964, p 30). The matrix **P** is formed by arranging the $2n$ vectors $(\mathbf{P}_\nu)$ $(\nu = 1 \ldots 2\mathrm{n})$ side by side. It will be shown that the matrix **P** as determined from (2.21) is canonical. To accomplish this, one considers two equations (2.21) for two distinct roots λ_ν and λ_μ:

$$\mathbf{H}(\boldsymbol{P}_\nu) = \lambda_\nu \tilde{\mathbf{K}}(\boldsymbol{P}_\mu) \tag{2.22a}$$

and

$$\mathbf{H}(\boldsymbol{P}_\mu) = \lambda_\mu \tilde{\mathbf{K}}(\boldsymbol{P}_\mu). \tag{2.22b}$$

Equation (2.22*a*) is multiplied from the left by the transpose of the vector $(\tilde{\boldsymbol{P}}_\mu)$, which is a row vector, and similarly equation (2.22*b*) is multiplied by $(\tilde{\boldsymbol{P}}_\nu)$ also from the left. Because matrix **H** is symmetric, the indices ν and μ can be interchanged in the product on the left without changing the sign; matrix **K** is skew-symmetric and, on interchanging indices ν and μ, the sign on the right side changes. So the indices in one of the equations are interchanged and this equation is subtracted from the other; the result is

$$(\lambda_\nu + \lambda_\mu)(\tilde{\boldsymbol{P}}_\mu)\tilde{\mathbf{K}}(\boldsymbol{P}_\nu) = 0. \tag{2.23}$$

The second factor in equation (2.23) can be different from zero only if $(\lambda_\nu + \lambda_\mu)$ is zero, and this is the case according to the arrangement of the roots (2.20) when $|\nu-\mu|=n$. The product $(\tilde{\boldsymbol{P}}_\mu)\tilde{\mathbf{K}}(\boldsymbol{P}_\nu)$ in equation (2.23) is in fact the matrix elements $(\tilde{P}\tilde{K}P)_{\mu\nu}$; consequently one can conclude that

$$(\tilde{P}\tilde{K}P)_{\mu\nu}=\begin{cases}A_{\mu\nu}; & \text{for } |\mu-\nu|=n\\ 0, & \text{otherwise.}\end{cases} \tag{2.24}$$

where μ, $\nu=1\ldots 2n$. The unspecified constants $A_{\mu\nu}$ must all be different from zero, otherwise the matrix $\tilde{\mathbf{P}}\tilde{\mathbf{K}}\mathbf{P}$ will be singular, implying linear dependence of columns of the matrix $\mathbf{P}$, contrary to the theorem of linearly dependent solutions of (2.21). These unspecified constants $A_{\mu\nu}$ are determined so that

$$\tilde{\mathbf{P}}\tilde{\mathbf{K}}\mathbf{P}=\tilde{\mathbf{K}}. \tag{2.25}$$

It is a simple matter to show that matrix $\mathbf{P}$ satisfying equation (2.25) also satisfies

$$\mathbf{P}\mathbf{K}\tilde{\mathbf{P}}=\mathbf{K} \tag{2.26}$$

and it will be omitted.

To pseudo-diagonalise the Hamiltonian matrix one multiplies equation (2.16) by $\tilde{\mathbf{P}}$ from the left, and using equation (2.25) one finds

$$\tilde{\mathbf{P}}\mathbf{H}\mathbf{P}=\tilde{\mathbf{K}}\mathbf{D}. \tag{2.27}$$

Due to the arrangement of the roots (2.20), matrix $\mathbf{D}$ in equation (2.27) has the form

$$\mathbf{D}=\begin{pmatrix}\mathbf{D}_1 & 0\\ 0 & -\mathbf{D}_1\end{pmatrix} \tag{2.28}$$

and

$$\mathbf{K}\mathbf{D}=\begin{pmatrix}0 & \mathbf{D}_1\\ \mathbf{D}_1 & 0\end{pmatrix}, \tag{2.29}$$

where the elements of the diagonal matrix D_1 are $\lambda_1\ldots\lambda_n$.

2.3. *Diagonalisation of the Hamiltonian matrix*

In § 2.2 it has been shown that the canonical matrix $\mathbf{P}$ as determined from equations (2.21) and (2.25) brings the Hamiltonian matrix $\mathbf{H}$ (2.9) into pseudo-diagonal form:

$$\tilde{\mathbf{P}}\mathbf{H}\mathbf{P}=\mathbf{H}^1=\begin{pmatrix}0 & \mathbf{D}_1\\ \mathbf{D}_1 & 0\end{pmatrix}. \tag{2.30}$$

The matrix $\mathbf{H}$ is then transformed into the diagonal form by a canonical matrix $\mathbf{T}$:

$$\mathbf{T}=(1/2^{1/2})\begin{pmatrix}\mathbf{D}_\alpha & \mathbf{D}_b\\ \mathbf{D}_c & \mathbf{D}_d\end{pmatrix} \tag{2.31a}$$

where $\mathbf{D}_\alpha$, $\mathbf{D}_b$, $\mathbf{D}_c$ and $\mathbf{D}_d$ and $n\times n$ diagonal matrices with elements: $a_1\ldots a_n$, $b_1\ldots b_n$, $c_1\ldots c_n$ and $d_1\ldots d_n$ respectively. These elements are linked through the relations

$$\tfrac{1}{2}(a_\alpha d_\alpha+b_\alpha c_\alpha)=0 \tag{2.31b}$$

and

$$\tfrac{1}{2}(a_\alpha d_\alpha-b_\alpha c_\alpha)=1. \tag{2.31c}$$

Alternatively,

$$a_\alpha d_\alpha = 1 \tag{2.31d}$$

and

$$b_\alpha c_\alpha = -1 \qquad \alpha = 1 \ldots n. \tag{2.31e}$$

The relation (2.31*b*) ensures that the transformation by the matrix **T** diagonalises the Hamiltonian matrix $\mathbf{H}^1$ (equation (2.30)):

$$\tilde{\mathbf{T}}\mathbf{H}^1\mathbf{T} = \mathbf{H}^{(2)} = \begin{pmatrix} \mathbf{D}_2 & 0 \\ 0 & \mathbf{D}_3 \end{pmatrix} \tag{2.32}$$

where the diagonal matrices $\mathbf{D}_2$ and $\mathbf{D}_3$ have elements $a_1c_1\lambda_1 \ldots a_nc_n\lambda_n$ and $b_1d_1\lambda_1 \ldots b_nd_n\lambda_n$, respectively. The condition (2.31*c*) ensures that the matrix **T** is canonical. Therefore the Hamiltonian matrix **H** (2.9) is diagonalised by the canonical matrix **F**,

$$\mathbf{F} = \mathbf{PT} \tag{2.33}$$

where matrix **P** is determined by following the procedure outlined by equations (2.17) or (2.21) and (2.25), and matrix **T** is given by (2.31*a*).

The Hamiltonian matrix $\mathbf{H}^{(2)}$ (equation (2.32)) can be further simplified to correspond to the sum of the Hamiltonians in the usual form $p^2+\omega^2x^2$ by a scale transformation of coordinates associated with the matrix $\mathbf{H}^{(2)}$, say $x^{(2)}$ and $p^{(2)}$:

$$x_\alpha^{(2)} \to (a_\alpha c_\alpha \lambda_\alpha)^{1/2} x_\alpha^{(2)} \qquad \alpha = 1 \ldots n$$

or, to be consistent with the matrix notation, by transforming the Hamiltonian matrix $\mathbf{H}^{(2)}$ by the matrix **S** into the form

$$\tilde{\mathbf{S}}\mathbf{H}^{(2)}\mathbf{S} = \bar{\mathbf{H}} = \begin{pmatrix} \mathbf{D}_\lambda & 0 \\ 0 & \mathbf{I} \end{pmatrix} \tag{2.34}$$

where the elements of the diagonal matrix $\mathbf{D}_\lambda$ are $-\lambda_1^2 \ldots -\lambda_n^2$ and the relations (2.31*d*) and (2.31*e*) have been used. The diagonal matrix **S** has the elements $(b_1d_1\lambda_1)^{1/2} \ldots (b_nd_n\lambda_n)^{1/2}, (b_1d_1\lambda_1)^{-1/2} \ldots (b_nd_n\lambda_n)^{-1/2}$.

The summary of all these transformations is as follows. The original Hamiltonian (2.7) is a function of $\boldsymbol{x}$ and $\boldsymbol{p}$. The linear canonical transformation of the old coordinates $\boldsymbol{x}$ and momenta $\boldsymbol{p}$ into the new coordinates $\bar{\boldsymbol{x}}$ and momenta $\bar{\boldsymbol{p}}$ is effected by the matrix **PTS**:

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix} = \mathbf{PTS}\begin{pmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{p}} \end{pmatrix} \tag{2.35}$$

and the original Hamiltonian (2.7) is transformed into the form of n uncoupled harmonic oscillators:

$$\bar{\mathscr{H}} = \sum_{\alpha=1}^{n} (\bar{p}_\alpha^2 - \lambda_\alpha^2 \bar{x}^2) = \sum_{\alpha=1}^{n} \bar{H}_\alpha. \tag{2.36}$$

This form corresponds to the Hamiltonian matrix $\bar{H}$ (equation (2.34)). The individual Hamiltonians in equation (2.36) are classified according to the value of a root λ_β.

For λ_β purely imaginary, the Hamiltonian $\bar{H}_\beta$ is one of the attractive harmonic oscillators. This Hamiltonian is related to bound states of the spectrum of the original Hamiltonian.

For λ_γ real, the Hamiltonian $\bar{H}_\gamma$ is one of the repulsive harmonic oscillator. This is an unbounded Hamiltonian and is related to unbounded or scattered states of the original Hamiltonian.

For λ_δ complex, the associated Hamiltonian $\bar{H}_\delta$ and the Hamiltonian associated with the complex conjugate root λ_δ^* (if λ is a root then λ^* is also a root) together can be canonically transformed into the real form. If the coordinates associated with the complex root λ_δ are $\bar{x}_1$ and $\bar{p}_1$ and those associated with the complex conjugate root are $\bar{x}_2$ and $\bar{p}_2$, then the replacement

$$\begin{aligned}
&\bar{x}_1 \rightarrow (2\lambda_\delta)^{-1/2}(\bar{x}_1 + \mathrm{i}\bar{p}_2)\\
&\bar{x}_2 \rightarrow (2\lambda_\delta^*)^{-1/2}(\bar{x}_2 + \mathrm{i}\bar{p}_1)\\
&\bar{p}_1 \rightarrow (\lambda/2)^{1/2}(\mathrm{i}\bar{x}_2 + \bar{p}_1)\\
&\bar{p}_2 \rightarrow (\lambda_{\delta/2}^*)^{1/2}(\mathrm{i}\bar{x}_1 + \bar{p}_2)
\end{aligned}\tag{2.37}$$

brings the Hamiltonian

$$H_\delta = (\bar{p}_1^2 - \lambda_\delta^2\bar{x}_1^2) + (\bar{p}_2^2 - \lambda_\delta^{*2}\bar{x}_2^2)\tag{2.38}$$

into the form

$$\mathrm{Re}(\lambda_\delta)(\bar{p}_1^2 + \bar{p}_2^2) - 2\,\mathrm{Im}(\lambda_\delta)(\bar{x}_1\bar{p}_2 - \bar{x}_2\bar{p}_1) - \mathrm{Re}(\lambda_\delta)(\bar{x}_1^2 + \bar{x}_2^2).\tag{2.39}$$

After further scale transformation,

$$\begin{aligned}
&\bar{x}_1 \rightarrow [\mathrm{Re}(\lambda_\delta)]^{1/2}\bar{x}_1\\
&\bar{x}_2 \rightarrow [\mathrm{Re}(\lambda_\delta)]^{1/2}\bar{x}_2\\
&\bar{p}_1 \rightarrow [\mathrm{Re}(\lambda_\delta)]^{-1/2}\bar{p}_1\\
&\bar{p}_2 \rightarrow [\mathrm{Re}(\lambda_\delta)]^{-1/2}\bar{p}_2
\end{aligned}$$

the Hamiltonian becomes

$$\mathcal{H}_\delta = \bar{p}_1^2 + \bar{p}_2^2 - 2\,\mathrm{Im}(\lambda_\delta)L_{12} - (\mathrm{Re}\,\lambda_\delta)^2(\bar{x}_1^2 + \bar{x}_2^2)\tag{2.40}$$

where the angular momentum is $L_{12} = \bar{x}_1\bar{p}_2 - \bar{x}_2\bar{p}_1$.

The Hamiltonian (2.40) is one of a particle moving in the plane in the repulsive harmonic oscillator potential, so it is again related to the unbounded part of the spectrum of the original Hamiltonian (2.7). Nevertheless, the Hamiltonian (2.40) does not seem to be well known and its properties will require further investigation.

The appearance of unbounded Hamiltonians in the transformed form (2.36) is a consequence of generality of the original Hamiltonian. If, for example, the original Hamiltonian is unbounded, i.e. in the repulsive oscillator form, it cannot be transformed, at least by this method, into the attractive oscillator Hamiltonian. A canonical transformation preserves equations of motion of the system and the motion obviously differs for attractive and repulsive harmonic oscillators.

The procedure for diagonalising the Hamiltonian matrix is analogous to the preocedure for diagonalising symmetric matrices in ordinary space. If the symplectic matrix $\mathbf{K}$ is replaced by a unit matrix the two procedures will be identical. This is in agreement with the remark of McIntosh (1975) about the similarities of the two spaces.

3. An application

As an illustration of the method, the Hamiltonian of a harmonic oscillator in a constant magnetic field is transformed into the Hamiltonian of the two-dimensional, in general, anisotropic oscillator. The same Hamiltonian has been studied by Dulock and McIntosh (1966) by using a special technique, and it is found that pertinent results of their work are identical to those obtained in this section.

The Hamiltonian of a harmonic oscillator in a magnetic field is

$$\mathscr{H}_{\mathrm{M}} = (1/2m)[\boldsymbol{p} - (e/c)\boldsymbol{A}]^2 + \tfrac{1}{2}m\omega^2 r^2 \tag{3.1}$$

where m and e are mass and charge of the particle, $\boldsymbol{p}$ and $\boldsymbol{r}$ are the two-dimensional momenta and position vector, c is the velocity of light in vacuum, ω is the 'natural' frequency of the oscillator, and $\boldsymbol{A}$ is a vector potential of the magnetic field.

The magnetic field is assumed to be constant and to point along the positive z axis. The symmetric gauge is chosen so that the vector potential has the components

$$\boldsymbol{A}: (-\tfrac{1}{2}B_0 x_2, \tfrac{1}{2}B_0 x_1, 0)$$

where B_0 is the magnetic field strength. To simplify the notation, the following constants are introduced:

$$\alpha = (e^2 B_0^2/8mc^2) + (m\omega^2/2), \tag{3.2a}$$

$$\beta = eB_0/2mc \tag{3.2b}$$

and

$$\gamma = 1/2m. \tag{3.2c}$$

The symmetrised Hamiltonian in the sense of equation (2.8) is

$$\mathscr{H}_{\mathrm{M}} = \alpha(x_1^2 + x_2^2) - (\beta/2)(x_1 p_2 - x_2 p_1) - (\beta/2)(p_2 x_1 - p_1 x_2) + \gamma(p_1^2 + p_2^2) \tag{3.3}$$

and, accordingly, the Hamiltonian matrix is

$$\mathbf{H}_{\mathrm{M}} = \begin{pmatrix} \alpha & 0 & 0 & -\beta/2 \\ 0 & \alpha & \beta/2 & 0 \\ 0 & \beta/2 & \gamma & 0 \\ -\beta/2 & 0 & 0 & \gamma \end{pmatrix}. \tag{3.4}$$

The first step is to solve the pseudosecular equation

$$\mathrm{d}(\mathbf{H}_{\mathrm{M}} - \lambda\tilde{\mathbf{K}}) = 0. \tag{3.5}$$

The four roots of equation (3.4) are symmetrically distributed around zero:

$$\pm \mathrm{i}a_+ \quad \text{and} \quad \pm \mathrm{i}a_- \tag{3.6}$$

where

$$a_\pm = \beta/2 \pm (\alpha\gamma)^{1/2}.$$

The second step is to find four columns of the matrix $\mathbf{P}_{\mathrm{M}}$ from

$$\mathbf{H}_{\mathrm{M}}(\mathbf{P}_\alpha) = \lambda_\alpha \tilde{\mathbf{K}}(\mathbf{P}_\alpha) \tag{3.7}$$

by successively substituting the four roots (3.6). The matrix $\mathbf{P}_M$ thus determined has the form

$$\mathbf{P}_M = \begin{pmatrix} -\sigma w_1 & \sigma w_2 & -\sigma w_3 & \sigma w_4 \\ -i\sigma w_1 & i\sigma w_2 & i\sigma w_3 & -i\sigma w_4 \\ -iw_1 & -iw_2 & iw_3 & iw_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix} \tag{3.8}$$

where $\sigma = (\gamma/\alpha)$ and w_1, w_2, w_3 and w_4 are undetermined parameters linked through the relations

$$4iw_1w_3 = -1 \tag{3.9a}$$

and

$$4iw_2w_4 = 1. \tag{3.9b}$$

The relations (3.9) ensure the canonicity of the matrix $\mathbf{P}$.

The matrix $\mathbf{P}$ transforms the Hamiltonian matrix (3.4) into a pseudodiagonal form

$$\tilde{\mathbf{P}}_M\mathbf{H}_M\mathbf{P}_M = \mathbf{H}_M^{(1)} = \begin{pmatrix} 0 & 0 & ia_+ & 0 \\ 0 & 0 & 0 & ia_- \\ ia_+ & 0 & 0 & 0 \\ 0 & ia_- & 0 & 0 \end{pmatrix}. \tag{3.10}$$

The third step is to transform the Hamiltonian matrix $\mathbf{H}_M^{(1)}$ into a diagonal form by the matrix $\mathbf{T}_M$ (the elements of the matrix $\mathbf{T}$, $\mathbf{D}_a = \mathbf{D}_b = -\mathbf{D}_c = \mathbf{D}_d = \mathbf{I}$, are chosen in order to simplify notation):

$$\mathbf{T}_M = 2^{-1/2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{3.11}$$

$$\tilde{\mathbf{T}}_M\mathbf{H}_M^{(1)}\mathbf{T}_M = \mathbf{H}^{(2)} = \begin{pmatrix} -ia_+ & 0 & 0 & 0 \\ 0 & -ia_- & 0 & 0 \\ 0 & 0 & ia_+ & 0 \\ 0 & 0 & 0 & ia_- \end{pmatrix}. \tag{3.12}$$

The fourth step is to effect a simple scale transformation of coordinates associated with the matrix $\mathbf{H}_M^{(2)}$. This transformation is represented by the matrix $\mathbf{S}$

$$\mathbf{S}_M = \begin{pmatrix} (ia_+)^{1/2} & 0 & 0 & 0 \\ 0 & (ia_-)^{1/2} & 0 & 0 \\ 0 & 0 & (ia_+)^{-1/2} & 0 \\ 0 & 0 & 0 & (ia_-)^{-1/2} \end{pmatrix}. \tag{3.13}$$

Finally, the Hamiltonian matrix is in the harmonic oscillator form,

$$\tilde{\mathbf{S}}_M\mathbf{H}_M^{(2)}\mathbf{S}_M = \bar{\mathbf{H}}_M = \begin{pmatrix} a_+^2 & 0 & 0 & 0 \\ 0 & a_-^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.14}$$

Therefore the coordinate transformation

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix} = \mathbf{P}_{\mathrm{M}}\mathbf{T}_{\mathrm{M}}\mathbf{S}_{\mathrm{M}} \begin{pmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{p}} \end{pmatrix} \tag{3.15}$$

brings the Hamiltonian (3.3) into the form:

$$\bar{\mathscr{H}}_{\mathrm{M}} = \bar{p}_1^2 + \bar{p}_2^2 + a_+^2\bar{x}_1^2 + a_-^2\bar{x}_2^2. \tag{3.16}$$

This is the Hamiltonian of the two-dimensional anisotropic oscillator, and the well known group-theoretical results can be applied immediately. The symmetry group of the Hamiltonian (3.16) is SU(2). For the ratio of frequencies

$$\frac{a_-}{a_+} = \frac{\beta/2 - (\alpha\gamma)^{1/2}}{\beta/2 + (\alpha\gamma)^{1/2}}$$

an irrational number, generators of SU(2) are transcendental functions of $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{p}}$. The 'simple' constants of motion of the Hamiltonian are obviously the two operators

$$\bar{C}_1 = \bar{p}_1^2 + a_+^2\bar{x}_1^2 \tag{3.17a}$$

and

$$\bar{C}_2 = \bar{p}_2^2 + a_-^2\bar{x}_2. \tag{3.17b}$$

The same operators are constants of motion of the original Hamiltonian (3.3). When these operators are transformed back into the original coordinates $\boldsymbol{x}$ and $\boldsymbol{p}$ with the matrix

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{p}_1 \\ \bar{p}_2 \end{pmatrix} = (\mathbf{P}_{\mathrm{M}}\mathbf{T}_{\mathrm{M}}\mathbf{S}_{\mathrm{M}})^{-1} \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix} \tag{3.18}$$

they are

$$-(2\sqrt{2}a_+a_-)^{-1}(a_-\bar{C}_1 + a_+\bar{C}_2) = C_1 = x_1p_2 = p_2x_1 \tag{3.19a}$$

and

$$(\sqrt{2}\,\delta a_+a_-)^{-1}(a_-\bar{C}_1 - a_+\bar{C}_2) = C_2 = p_1^2 + p_2^2 + \delta^{-2}(x_1^2 + x_2^2) \tag{3.19b}$$

where the irrelevant proportionality constant has been eliminated. The first constant of motion is the angular momentum of the particle and the second constant is related to the total energy through the linear combination

$$\mathscr{H}_{\mathrm{M}} = \gamma C_2 - \beta C_1. \tag{3.20}$$

In this way the relevant results of Dulock and McIntosh—the symmetry group is SU(2), the ratio of frequencies, constants of motion and the form of the Hamiltonian in terms of the constants of motion—have been reproduced by systematically applying the method of this paper.

4. Concluding remarks

The method developed in § 2 for canonically transforming the general quadratic

Hamiltonian is applicable only in a 'clean' case, i.e. if none of the roots of (2.18) are zero and if there is no degeneracy among the roots.

Also it may happen that some of the elements of the matrix **P** are complex. This means that the new coordinate and momentum operators are no longer Hermitian. This possibility has been considered by Wolf (1974a, b), Boyer and Wolf (1975) and Kramers *et al* (1975). Of course, non-Hermiticity of these operators does not affect symmetry considerations.

In recent publications (Leach 1977, 1978), a problem closely related to this has been considered. It has been shown that the quadratic Hamiltonians of the type (2.7) can be transformed into the form of attractive harmonic oscillators by a time-dependent canonical transformation. The actual determination of the canonical matrix requires solving $2n \times 2n$ coupled differential equations of the first order. The Hamiltonian arrived at is one of the isotropic attractive harmonic oscillators, unlike the transformed Hamiltonian of this paper, which may contain repulsive harmonic oscillator terms. The canonical transformation of the general Hamiltonian (2.7) into the Hamiltonian of the uncoupled harmonic oscillators is not unique. One of the three factors of the transformation, the matrix **T**, depends upon $4n$ parameters with $2n$ relations between them, leaving $2n$ unspecified parameters. It can be shown that the set of all these $2n$ parameter-dependent canonical transformations, all of which transform the original Hamiltonian into the same form, constitute a $2n$ parameter group, actually the subgroup of the full symplectic group $\mathrm{Sp}(2n)$, and this group is the symmetry group of the transformed Hamiltonian as well as of the original one.

Because a canonical transformation preserves the commutational relations between coordinate and momentum operators, it preserves the symmetry and dynamical symmetry of the original Hamiltonian as well as the Heisenberg equation of motion for the operators which are functions of coordinate and momentum operators. So the method can be used to study the group-theoretical structure of the quadratic Hamiltonians by transforming them into a well understood harmonic oscillator form, as has been done in § 3 with the Hamiltonian of the harmonic oscillator in a constant magnetic field.

Analyses of the repulsive type of Hamiltonians which appear as terms of the transformed Hamiltonian (2.36), the group-theoretical analysis of the quadratic Hamiltonians in relation to their transformed form and relation of this method to one proposed by Leach (1977, 1978) are planned to be the subjects of future studies.

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